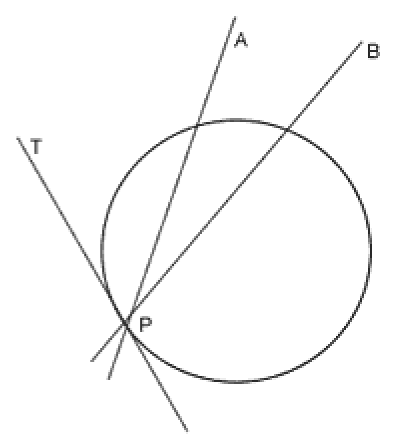
Circle

- Circle is a closed figure of points which are at a constant distance from a fixed points, centre and constant distance is called radius.
- Two circles are congruent if and only if they have equal radii.
- Equal chords of a circle subtend equal angles at the centre and conversely.
- Two arcs of a circle are congruent if their corresponding chords are equal and conversely.
- The perpendicular drawn from the centre of a circle to a chord bisects the chord.
- The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.

Tangent to a Circle

- Tangent to a circle is a line which intersects the circle in exactly one point.

- At a point of a circle there is one and only one tangent.
- The tangent at any point of a circle is perpendicular to the radius through the point of contact.
- The lengths of tangents drawn from an external point to a circle are equal.
- Centre of the circle lies on the bisector of the angle between the two tangents.



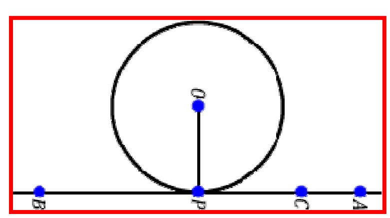
Theorem 1

The tangent at any point on a circle is perpendicular to the radius drawn to the point of contact.

Given: A tangent AB with point of contact P.

To prove: OP $\perp$ AB

Proof:



Consider point C on AB other than P.

C must lie outside the circle. ($\because$ A tangent can have only one point of contact with the circle)

OC > OP ($\because$ C lies outside the circle)

This is true for all positions of C on AB.

Thus, OP is the shortest distance between point P and line segment AB.

Hence, OP $\perp$ AB.

Theorem 2

Tangents drawn to a circle from an external point are equal in length.

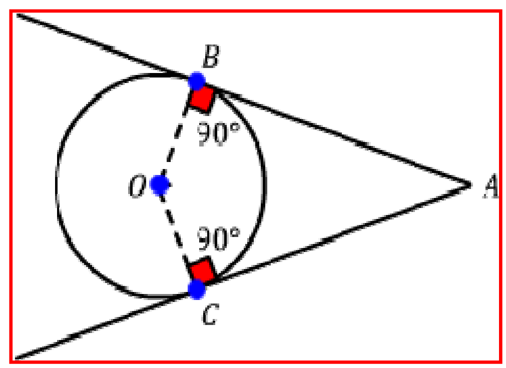
Given:

Two tangents AB and AC from an external point A to points B and C on a circle

To prove: AB = AC

Construction: Join OA, OB and OC.

Proof:



In triangles OAB and OAC,

$\angle OBA = 90^o$ (Radius OB $\perp$ Tangent AB at B)
$\angle OCA = 90^o$ (Radius OC $\perp$ Tangent AC at C)
In triangles OBA and OCA,

$\angle OBA = \angle OCA = 90^o$

OB = OC (Radii of the same circle)

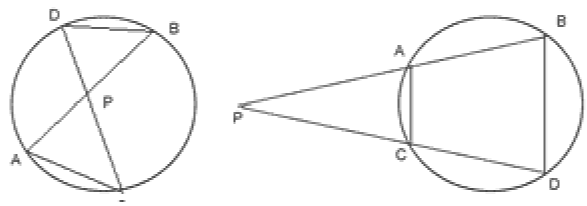
OA = OA (Common side)

Thus, $\Delta OBA \cong \Delta OCA$ (RHS congruence rule)

Hence, AB = AC (Corresponding sides of congruent triangles)

Theorem 3

If two chords of a circle intersect inside or outside the circle, then the rectangle formed by the two parts of one chord is equal in area to the rectangle formed by the two parts of the other.



Given

Two chords AB and CD of a circle such that they intersect each other at a point P lying inside in figure (i) or outside in figure (ii) of the circle.

To prove: - PA.PB = PC.PD

Construction:-AC and BD are joined P.

Proof:-

Case - (1) in figure (i) P lies inside the circle

In $\Delta^{S} PCA \, and \, PBD,$ we have

$\angle PCA = \angle PBD$ [Angles in jthe same segment]

$\angle APC = \angle BPD$ [vertically opposite angles]

$\therefore \Delta PCA \sim \Delta PBD$ (AA similarity)

Case- (2) In figure (ii) P lies outside the circle

$\angle PAC + \angle CAB = 180^{0}$ (limear pair)

and $\angle CAB + \angle PDB = 180^{0}$ (opposite angles of acyclicq and)

$\therefore \angle PAC = \angle PDB$

In $\Delta^s PCA \text{ and } PBD$

$\angle PAC = \angle PDB$ [Proved above]

$\angle APC = \angle DPB$ [Common]

$\therefore \Delta PCA \sim \Delta PBD$ (AA similarity)

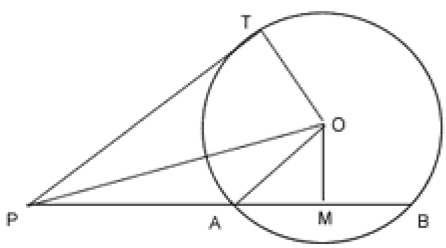
Hence, in either case,

$$\frac{PA}{PB} = \frac{PC}{PB}$$

Or, PA.PB = PC.PD

Theorem 4.

If PAB is a secant to a circle intersecting it at A and B and PT is a tangent then PA.PB = PT^2.



Given:

PAB is secant intersecting the circle with centre O at A and B and a tangent PT at T.

To Prove: - PA.PB = PT^2

Construction: - $OM \perp AB$ is drawn OA, OP and OT are joined.

Proof: - PA = PM – AM

PB = PM + MB

$$= PM = AM \qquad (\because AM = MB)$$

$$\therefore PA.PB = (PM - AM).(PM + AM)$$
$$= PM^2 - AM^2$$

Also $OM \perp AB$

$\therefore PM^2 = OP^2 - OM^2$ [Pythagoras theorem]

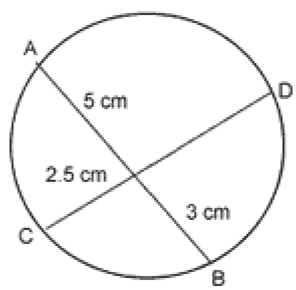
and $AM^2 = OA^2 - OM^2$ [Pythagoras theorem]

$$\begin{aligned}\therefore PA.PB &= PM^2 - AM^2 \\ &= (OP^2 - OM^2) - (OA^2 - OM^2) \\ &= OP^2 - OM^2 - OA^2 + OM^2 \\ &= OP^2 - OA^2 \\ &= OP^2 - OT^2 \qquad [\because OA = OT \text{ radii}]\end{aligned}$$

$\therefore PA.PB = PT^2$ [Pythagoras theorem.]

Example 1.

In figure, chords AB and CD of the circle intersect at O. OA = 5cm, OB = 3cm and OC = 2.5cm. Find OD.

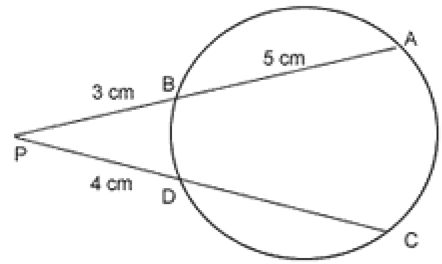


Solutions

Chords AB and CD of the circle intersect at O

$$\therefore OA \times OB = OC \times OD$$
$$Or, \quad 5 \times 3 = 2.5 \times OD$$
$$Or, \quad OD = \frac{2 \times 3}{2.5} = 6cm$$

Example 2. In figure. Chords AB and CD intersect at P.



If AB = 5cm, PB = 3cm and PD = 4cm. Find the length of CD.

Solution

PA = 5 + 3 = 8cm

PA X PB = PC X PD

Or, 8 X 3 = PC X 4

Or, PC = 8 X 3/4 = 6cm

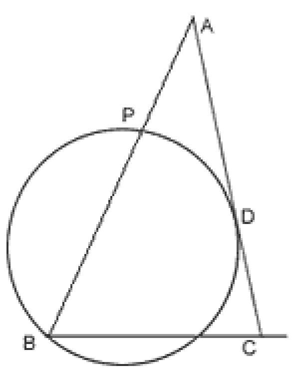
$\therefore$ CD = PC - PD

= 6 - 4

= 2cm

Example 3.

In the figure, ABC is an isosceles triangle in which AB = AC. A circle through B touches the side AC at D and intersects the side AB at P. If D is the midpoint of side AC, Then AB = 4AP.



Solution

AP X AB = AD^2 = $(1/2AC)^2$

$AP \times AB = 1/4\ AC^2$ [AD = 1/2AC]

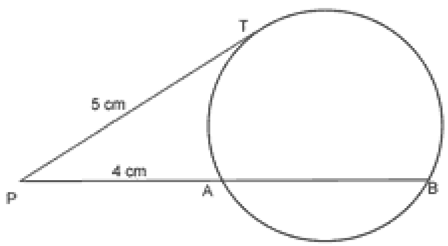
Or, $4\ AP.AB = AC^2$ [AC = AB]

Or, $4\ AP.AB = AB^2$

Or, 4 AP = AB

Example 4.

In the figure. Find the value of AB Where PT = 5cm and PA = 4cm.



Solution

$PT^2 = PA \times PB$ (Theory.2)

$5^2 = 4 \times PB$

PB = 25/4 = 6.25

AB = PB - PA

AB = 6.25 - 4

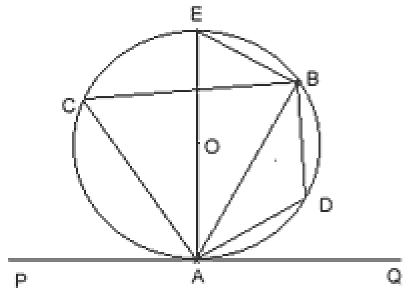
AB = 2.25 cm

Theorem 5.

If a line touches a circle and from the point of contact a chord is drawn, the angle which this chord makes with the given line are equal respectively to the angles formed in the corresponding alternate segments.

Given:

PQ is a tangent to circle with centre O at a point A, AB is a chord and C, D are points in the two segments of the circle formed by the chord AB.



To Prove:- (i) $\angle BAQ = \angle ACB$

(ii) $\angle BAP = \angle ADB$

Construction

A diameter AOE is drawn. BE is joined.

Proof: - In ΔABE

$\angle ABE = 90^0$ [Angle is a semicircle]

$\therefore \angle AEB + \angle BAE = 90^0$

$\angle BAE + \angle BAQ = \angle EAQ = 90^0$ $[EA \perp PQ]$

$\therefore AEB + \angle BAE = \angle BAE + \angle BAQ$

$\angle AEB = \angle BAQ$

We know that,

$\angle ACB = \angle AEB$

$\therefore \angle BAQ = \angle ACB$

Again $\angle BAQ + \angle BAP = 180^0$ [Linear Pair]

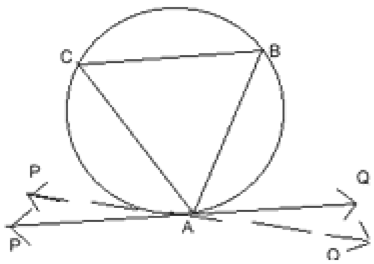
and $\angle ACB + \angle ADB = 180^0$ [Opposite angles of a cyclicquadric]

$\therefore \angle BAQ + \angle BAP = \angle ACB + \angle ADB$

$\angle BAP = \angle ADB$ $[\therefore \angle BAQ = \angle ACB]$

Theorem 6.

If a line is drawn through an end point of a chord of a circle so that the angle formed by it with the chord is equal to the angle subtend by chord in the alternate segment, then the line is a tangent to the circle.



Given:

A chord AB of a circle and a line PAQ. Such that $\angle BAQ = \angle ACB$ where c is any point in the alternate segment ACB.

To Prove:- PAQ is a tangent to the circle.

Construction:- Let PAQ is not a tangent then let us draw P' AQ' another tangent at A.

Proof: - AS P 'AQ' is tangent at A and AB is any chord

$\therefore \angle BAQ' = \angle ACB$ [theo.5]

But $\therefore \angle BAQ' = \angle ACB$ (given)

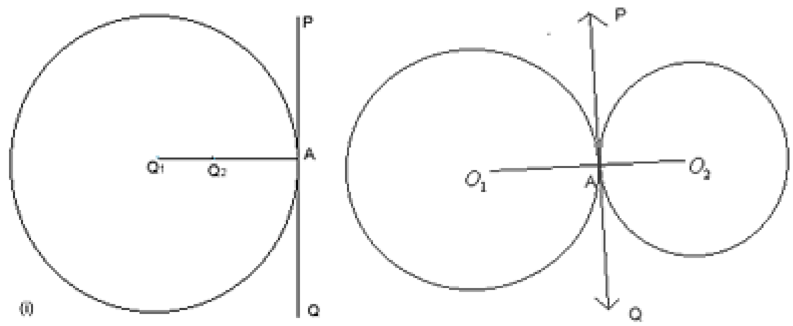
$\therefore \angle BAQ' = \angle BAQ$

Hence AQ' and AQ are the same line i.e. P' AQ' and PAQ are the same line.

Hence PAQ is a tangent to the circle at A.

Theorem 7.

If two circles touch each other internally or externally, the point of contact lie on the line joining their centres.



Given:

Two circles with centres O_1 and O_2 touch internally in figure (i) and externally in figure (ii) at A.

To prove: - The points O_1, O_2 and A lie on the same line.

Construction:- A common tangent PQ is drawn at A.

Proof: - In figure (i) $\angle PAQ_2 = \angle PAQ_1 = 90^0$ (PA is tangent to the two circles)

$\therefore$ O_1, O_2 and A are collinear.

In figure (ii) $\angle PAO_1 = \angle PAO_2 = 90^0$ (PA is tangent to the circles)

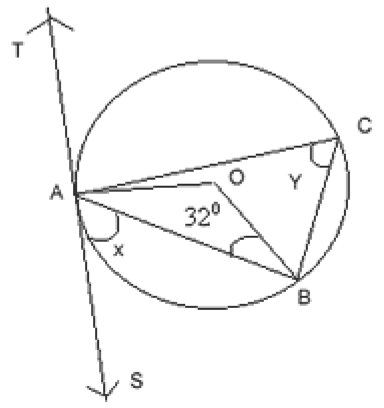
$$\therefore \angle PAQ_1 + \angle PAQ_2 = 90^0 + 90^0$$
$$= 180^0$$

i.e. $\therefore \angle PAO_1$ *and* $\angle PAO_2$ from a linear pair

$\therefore$ O_1, O_2 and A lie on the same line.

Example 5.

In the given figure TAS is a tangent to the circle, with centre O, at the point A. If $\angle OBA = 32^0$, find the value of x and y.



Solution:- $In \ \Delta OBA, OA = OB \qquad (radii)$

$\therefore \ \angle OAB = \angle OBA$

$= 32^0$

Now TAS is tangent at A

$\therefore \ OA \perp TAS$

$\therefore \ \angle OAB + x = 90^0$

$Or, \ \ 32^0 + x = 90^0$

$\therefore \ x = 58^0$

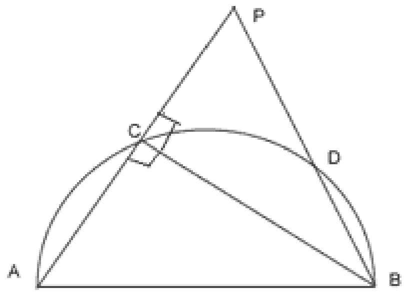
$\angle x = \angle y$ [Angles in the alternates]

$\angle x = \angle y = 58^0$

Example 6.

In the given figure. $\angle C$ is right angle of ΔABC,. A semicircle is drawn on AB as diameter. P is any point on AC produced. When joined, BP meets the semi-circle in point D.

Prove that: AB^2 = AC.AP + BD.BP.



Solution:-

$\angle ACB = 90^0$

$\therefore \ \angle BCP = 90^0$

$\therefore \ BC^2 = BP^2 - CP^2$

In ΔABC,

$$
\begin{aligned}
AB^2 &= AC^2 + BC^2 \\
&= AC^2 + BP^2 - CP^2 \\
&= AC^2 - CP^2 + BP^2 \\
&= (AC - CP)\ (AC + CP) + BP^2 \\
&= (AC - CP).AP + BP^2 \\
&= AP.AC - AP.CP + BP^2 \\
&= AP.AC + BP^2 - AP.CP \\
&= AP.AC + BP^2 - BP.PD \qquad [AP.CP = BP.PD] \\
&= AP.AC + BP(BP - PD) \\
&= AP.AC + BP.BD
\end{aligned}
$$

$\therefore \ AB^2 = AP.AC + BP.BD$