A Short Introduction to Matrices and Vectors

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1 Matrices

1.1 Definition

A *matrix* is a rectangular array of *objects*. (These objects will be real numbers in our applications, but they could be complex numbers, or even other matrices, if properly chosen.) There are two ways to designate a matrix:

1. As a set of members, where the subscript designates the position in the array: $\mathbf{A} = \{A_{ij}\}$ where $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots m$ and the matrix is said to have dimensions of n rows by m columns. $\mathbf{A}$ is the symbol representing the abstract matrix while A_{ij} is the symbol representing the object in the matrix at position ij.

2. As a displayed array:

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{bmatrix}$$

1.2 Operations

A set of matrices may be operated on to produce new matrices. The operations and the conditions which must be met for them to be value are:

multiplication by a scalar If s is a scalar object, that is the same kind of object as the elements of the matrix $\mathbf{A}$, then $\mathbf{B} = s\mathbf{A}$ where $B_{ij} = sA_{ij}$ for all i and j.

addition $\mathbf{C} = \mathbf{A} + \mathbf{B}$ when $C_{ij} = A_{ij} + B_{ij}$. All three matrices must have the same dimensions, n and m.

subtraction Subtraction means addition after multiplying the second term by the scalar -1. $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B}$.

multiplication A matrix can be multiplied by a matrix provided the number of columns in the matrix on the left in the product is equal to the number of rows in the matrix on the right in the product: $\mathbf{C}_{n\times m} = \mathbf{A}_{n\times p}\mathbf{B}_{p\times m}$ when $C_{ij} = \sum_{k=1}^{p} A_{ik}B_{kj}$. For example:

$$\begin{bmatrix} 6 & 22 \\ -9 & 24 \end{bmatrix} = \begin{bmatrix} 4 & 5 & -1 \\ 6 & 0 & 3 \end{bmatrix} \begin{bmatrix} -2 & 5 \\ 3 & 0 \\ 1 & -2 \end{bmatrix}$$

Note that $\mathbf{AB} \neq \mathbf{BA}$ in general, although they can be equal in special cases.

transpose The transpose of a matrix is the operation of interchanging rows and columns. It is symbolized by a “T” superscript: $\mathbf{A}^T$, where $A_{ij}^T = A_{ji}$:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

determinant of a square matrix The determinant of $\mathbf{A}$ is defined as the usual determinant of algebra on the square array of elements. This is symbolized as $|\mathbf{A}|$.

1.3 Special matrices

identity The identity for matrix multiplication is square, and has all diagonal elements the identities for multiplication of the matrix elements, and all other elements zeros. This special matrix is given the symbol **I**. $I_{ij} = \delta_{ij}$ where $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$ E.g., $\mathbf{I}_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

zero The zero matrix has all of its elements zero. There is no special symbol for the zero matrix. The scalar 0 is used for it. The 0 in a matrix equation is always assumed to have the dimensions required for the operation or equality. E.g.,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

inverse of a square matrix If a matrix is square, and if its determinant is not zero, then it has an inverse, symbolized by giving the original matrix symbol a -1 superscript, and defined by $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. E.g.,

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2 Vectors

2.1 Definitions and operations

vector A *vector* is an object having a magnitude and a direction. In three-dimensional space it takes three real numbers to define a vector. The symbol used to represent an abstract vector is a letter with a bar over it: $\overline{A}$ is a vector.

Note that when a vector represents a physical quantity the magnitude has a unit associated with it. The direction never has a unit. (In component form – see below – all components must have the same dimensionality, and usually all are expressed in the same units – all components given in, e.g., inches, not one component in inches and another in meters.)

equality of vectors Two vectors are said to be equal, e.g., $\overline{A} = \overline{B}$ if they have the same magnitude *and* the same direction. Equality does not depend on where in space the vectors are located or acting. A vector at one place may be equal to a vector at another place as long as both have the same magnitude and direction..

magnitude of a vector The magnitude of a vector is symbolized by $|\overline{A}| = A$, where $A \geq 0$.

direction – unit vectors The direction of a vector is symbolized by a *unit vector*, that is a vector having its magnitude equal to one, $\hat{A}$, where a "hat" symbol replaces the bar over the letter to indicate that the magnitude is one.

scalar A *scalar* is an object having a magnitude only. However, in this case, the magnitude may be signed – it is not an absolute value. Scalars are represented by ordinary letters: a.

multiplication of a vector by a scalar Multiplication of a scalar b times a vector $\overline{A}$ yields a new vector $\overline{C}$: $\overline{C} = b\overline{A}$. The direction of $\overline{C}$ is the same as the direction of $\overline{A}$ if $b > 0$ and direction of $\overline{C}$ is the opposite of the the direction of $\overline{A}$ if $b > 0$. The magnitude of $\overline{C}$ is $|\overline{C}| = |b||\overline{A}|$.

Hence to form a unit vector, one multiplies a vector by the reciprocal of its magnitude. This is often symbolized as a division by its magnitude, as division is understood to be multiplication by the reciprocal of the divisor. (The divisor must be a scalar. There is no definition of division of vectors by vectors.)

$\hat{A} = \overline{A}/|\overline{A}|$

zero vector A zero vector is one having magnitude zero. The direction can be any direction. The symbol is just the scalar symbol for zero: 0. Hence it is correct to write $\overline{A} = 0$ even though normally a vector can only be equated to a vector.

addition of vectors Addition of two vectors in the same direction is the vector in that direction that has the sum of the magnitudes of the two vectors. When two vectors are not in the same direction their sum is defined by the geometrical construction, using directed line segments to represent the vectors, that when the two are drawn with the tail of the second placed at the head of the first the sum is the line segment from the tail of the first to the head of the second. This is shown in Fig. 1.

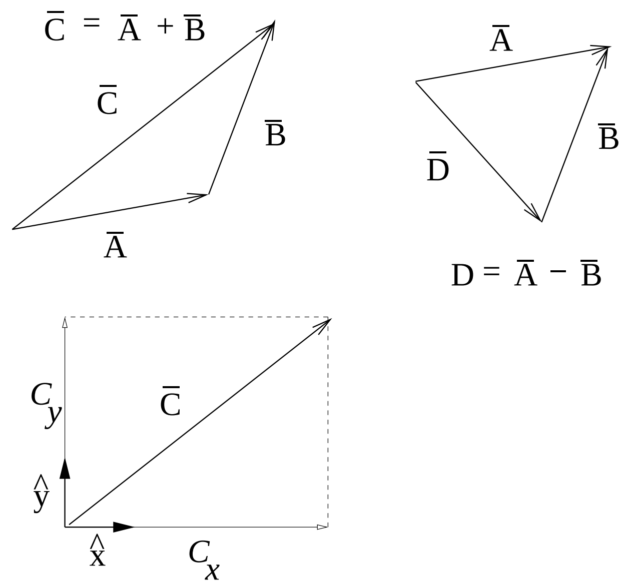

Figure 1: Addition and subtraction of vectors, and vector components.

component representation With the concepts of unit vectors, multiplication by a scalar, and addition given, one may represent any vector as a sum of three numbers multiplying three non-coplanar unit vectors. For this course will will take these three unit vectors to be parallel to the three Cartesian coordinate axes and give them symbols $\hat{x}$, $\hat{y}$, and $\hat{z}$. Then any vector can be written like this: $\overline{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$. The components A_x, A_y, and A_z are the projections along the coordinate axes of the line segment representing $\overline{A}$.

scalar multiplication of vectors Two vectors may be multiplied to form a scalar. This is called the scalar or dot product and is written $s = \overline{A} \cdot \overline{B}$. The scalar value s is defined as the product of the magnitudes of the two vectors times the cosine of the smallest angle between them. This angle may be visualized if the vectors are represented as directed line segments and both drawn from the starting point. The smallest angle is shown as θ in Fig. 2.

The dot products between the Cartesian unit vectors are
$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1$ and $\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0$.
Thus, in component form, $\overline{A} \cdot \overline{B} = A_xB_x + A_yB_y + A_zB_z$.

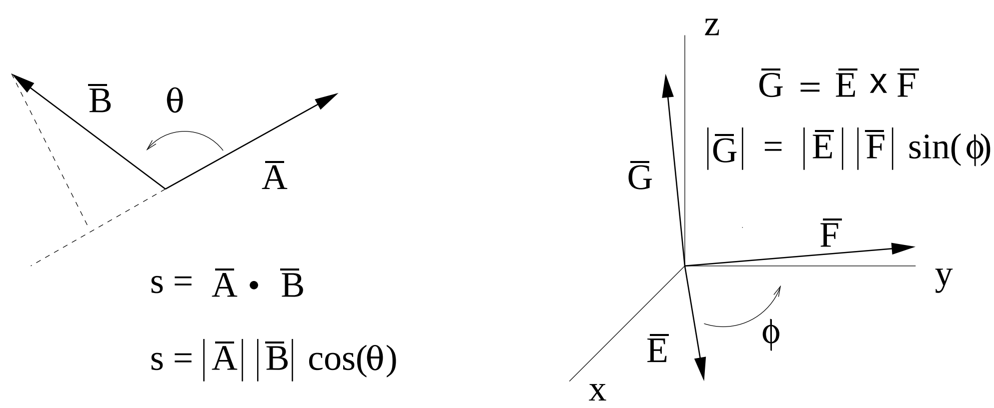

Figure 2: Multiplication of vectors.

vector multiplication of vectors Two vectors may be multiplied to get a third vector. The magnitude of this third vector is the product of the magnitudes of the two vectors times the sine of the smallest angle between them (as shown in Fig. 2). The direction for this third vector is taken to be in the only unique direction common to two vectors – it is perpendicular to the plane defined by these two vectors. (If the vectors are parallel, then this product has zero magnitude so its direction is unspecified.) Since there are two directions perpendicular to a plane, the one chosen is that given by a right-hand-rule. When the first vector is rotated through the smallest angle between them towards the 2nd vector, the direction which that motion would advance a right-handed screw is the direction of the vector product. Put another way, if the fingers of the right hand are pointed from the head of the first to the head of the second vector then the thumb points in the direction that is positive for the vector product. This is shown in Fig. 2.

The vector product is all called the cross product because of the way it is represented: $\overline{C} = \overline{A} \times \overline{B}$.

Because of the definition of the direction of $\overline{A} \times \overline{B}$ the result is negated when the order of the vectors in the product is reversed: $\overline{A} \times \overline{B} = -\overline{B} \times \overline{A}$.

The cross products between the Cartesian unit vectors are:
$\hat{x} \times \hat{x} = \hat{y} \times \hat{y} = \hat{z} \times \hat{z} = 0$, $\hat{x} \times \hat{y} = \hat{z}$, $\hat{y} \times \hat{z} = \hat{x}$, and $\hat{z} \times \hat{x} = \hat{y}$.

This allows the cross product to be written in component form as
$\overline{A} \times \overline{B} = \hat{x}(A_y B_z - A_z B_y) + \hat{y}(A_z B_x - A_x B_z) + \hat{z}(A_x B_y - A_y B_x)$.

This may also be written as a determinant: $\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$