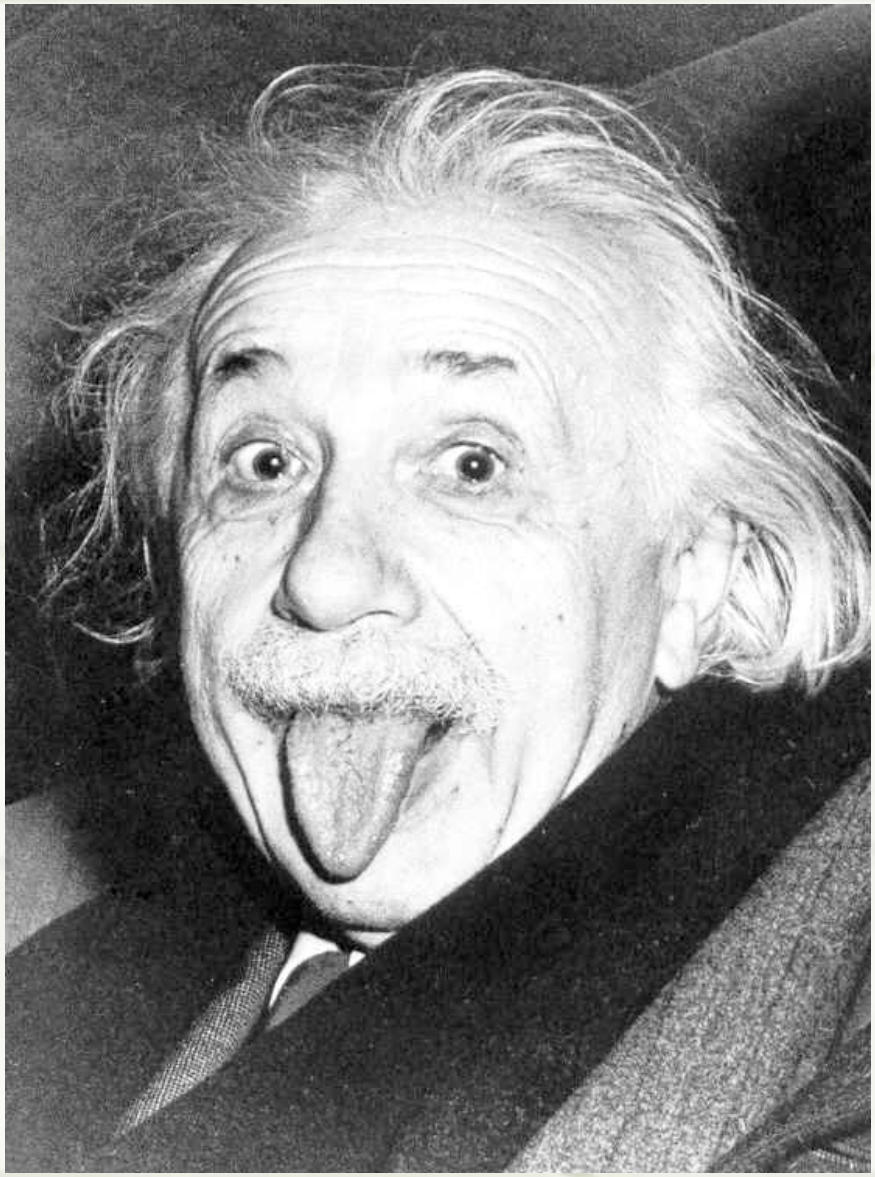
Derivation of $E=mc^2$



Yum-Yum Physics!

By the Work-Energy Theorem:

$$W = KE$$

In this case, we will consider *PE* = 0 so:

$$W = KE = E$$

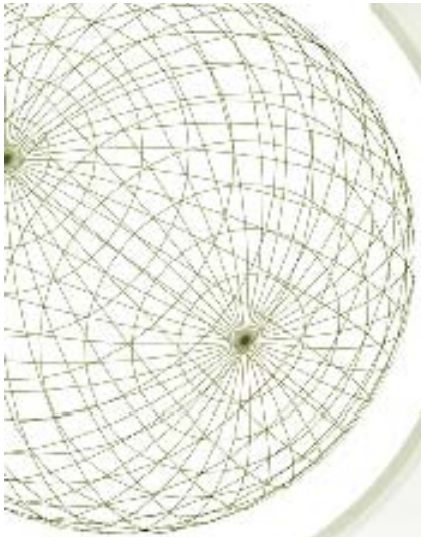
From the definition of work:

$$W = F(\Delta x)$$

$$E = F(\Delta x)$$

Therefore:

$$E = \int_0^x F\,dx$$



From Newton's original statement of *F=ma*:

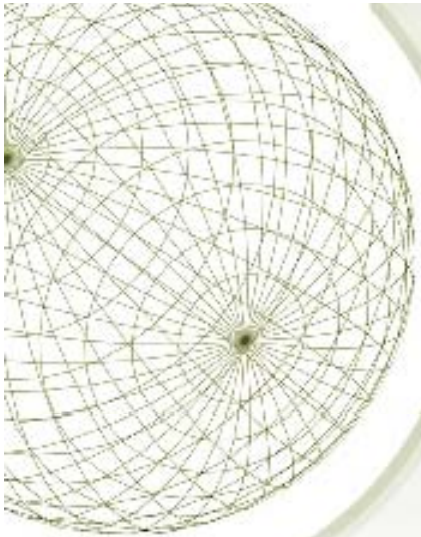
$$F = \frac{dm}{dt}v + m\frac{dv}{dt}$$

$$F = \frac{d}{dt}(mv)$$

This, along with the previous equation yields:

$$E = \int_0^x F\,dx$$

$$E = \int_0^x \frac{d}{dt}(mv)\,dx$$



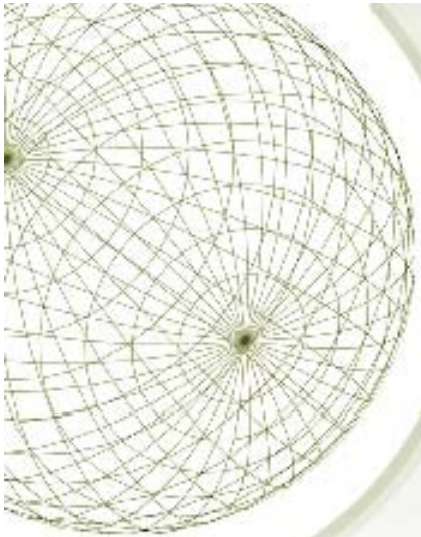
We have *dt* and *dx* in the equation.
Let's write it all in terms of *dt*:

$$v = \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

$$dx = v dt$$

When the variable changes from *dx* to *dt*,
we must also change the bounds of the integral

$$E = \int_0^x \frac{d}{dt}(mv)\,dx = \int_o^t \frac{d}{dt}(mv)v\,dt$$



We can eliminate the *dt*'s so:

$$E = \int_0^t \frac{d}{dt}(mv)v\,dt = \int_0^{mv} v\,d(mv)$$

Notice that the *dt* changed to *d*(*mv*) so the bounds also changed

For velocities approaching *c*, the mass increases

The relativistic mass is: $m = \dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$

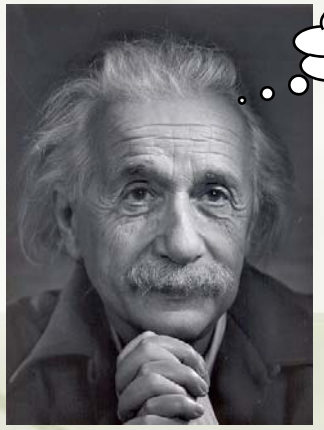
$$E = \int_0^{mv} v\,d(mv) = \int_0^{mv} v\,d\left(\frac{m_0 v}{\sqrt{1-\dfrac{v^2}{c^2}}}\right)$$

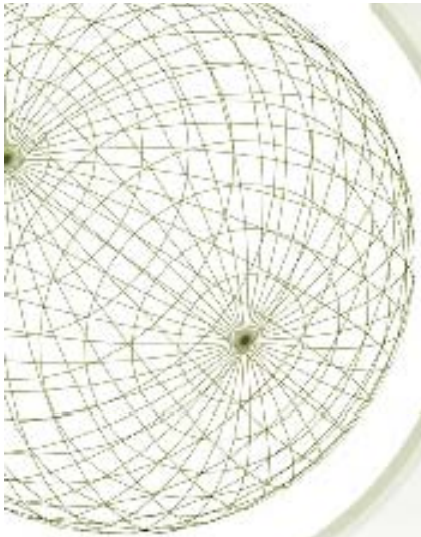
Since m_0 is a constant, it can factored out of the integral:

$$E = \int_0^{mv} v\,d\left(\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}\right) = m_0 \int_0^{v} v\,d\left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

Applying the Quotient Rule:
(keep in mind *v* is a variable and *c* is a constant)

$$d\left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{(dv)\sqrt{1-\frac{v^2}{c^2}} - v\left(\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\left(-\frac{1}{c^2}\right)(2v\,dv)}{\left(1-\frac{v^2}{c^2}\right)}$$





Combining these expressions for Energy yields:

$$E = m_0 \int_0^v v d\left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

$$E = m_0 \int_0^v v \frac{(dv)\sqrt{1-\frac{v^2}{c^2}} - v\left(\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\left(-\frac{1}{c^2}\right)(2vdv)}{\left(1-\frac{v^2}{c^2}\right)}$$

$$E = m_0 \int_0^v \left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{\frac{v^3}{c^2}}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\right) dv$$

To get a common denominator, we will multiply the numerator and denominator of $\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}$ by $\frac{(1-\frac{v^2}{c^2})}{(1-\frac{v^2}{c^2})}$

$$E = m_0 \int_0^v \left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{\frac{v^3}{c^2}}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}}\right) dv$$

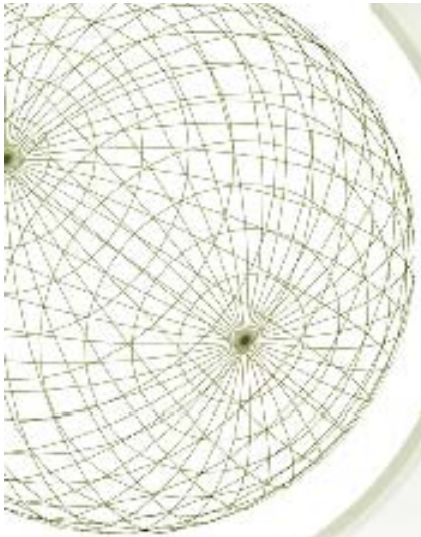
$$E = m_0 \int_0^v \left[\frac{(1-\frac{v^2}{c^2})v}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} + \frac{\frac{v^3}{c^2}}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}}\right] dv$$

Simplifying:

$$E = m_0 \int_0^v [\frac{(1-\frac{v^2}{c^2})v}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} + \frac{\frac{v^3}{c^2}}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}}]dv$$

$$E = m_0 \int_0^v \frac{v - \frac{v^3}{c^2} + \frac{v^3}{c^2}}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}}dv$$

$$E = m_0 \int_0^v \frac{v}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}}dv$$

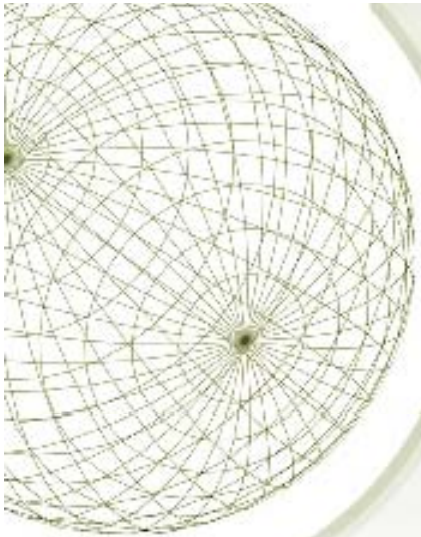


$$E = m_0 \int_0^v \frac{v}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}} dv$$

$$E = m_0 \int_0^v \frac{v}{(\frac{c^2}{c^2} - \frac{v^2}{c^2})^{\frac{3}{2}}} dv$$

$$E = m_0 \int_0^v \frac{v}{(\frac{c^2 - v^2}{c^2})^{\frac{3}{2}}} dv$$

$$E = m_0 \int_0^v \frac{v}{\frac{(c^2 - v^2)^{\frac{3}{2}}}{c^3}} dv$$



$$E = m_0 \int_0^v \frac{v}{\frac{(c^2 - v^2)^{\frac{3}{2}}}{c^3}} dv$$

$$E = m_0 \int_0^v \frac{c^3 v}{(c^2 - v^2)^{\frac{3}{2}}} dv$$

$$E = m_0 c^3 \int_0^v \frac{v}{(c^2 - v^2)^{\frac{3}{2}}} dv$$

To evaluate the integral we make the substitution:

$$u = c^2 - v^2$$

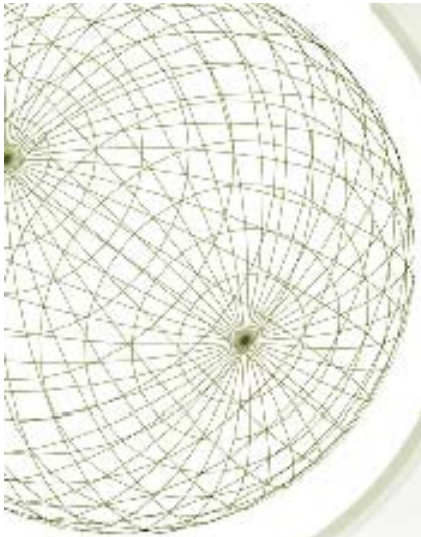
$$\frac{du}{dv} = -2vdv$$

$$-\frac{1}{2}du = vdv$$

Therefore:

$$E = m_0 c^3 \int_0^v \frac{vdv}{(c^2 - v^2)^{\frac{3}{2}}}$$

$$E = m_0 c^3 \int_{u(v=0)}^{u(v)} \frac{-\frac{1}{2}du}{u^{\frac{3}{2}}}$$

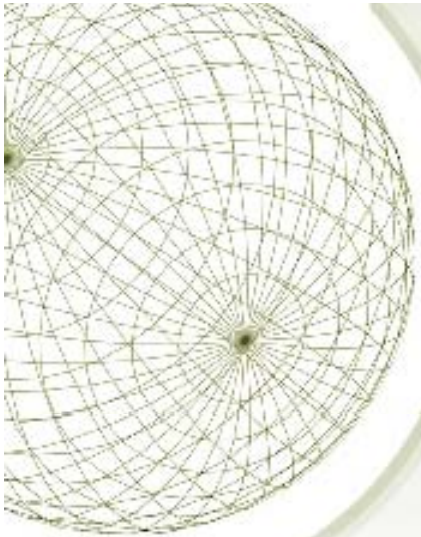


$$E = m_0 c^3 \int_{u(v=0)}^{u(v)} \frac{-\frac{1}{2} du}{u^{\frac{3}{2}}}$$

$$E = -\frac{m_0 c^3}{2} \int_{c^2}^{c^2 - v^2} \frac{du}{u^{\frac{3}{2}}}$$

$$E = -\frac{m_0 c^3}{2} \left(\frac{u^{-\frac{1}{2}}}{-\frac{1}{2}} \Big|_{c^2}^{c^2 - v^2} \right)$$

$$E = m_o c^3 \left(\frac{1}{\sqrt{c^2 - v^2}} - \frac{1}{\sqrt{c^2}} \right)$$

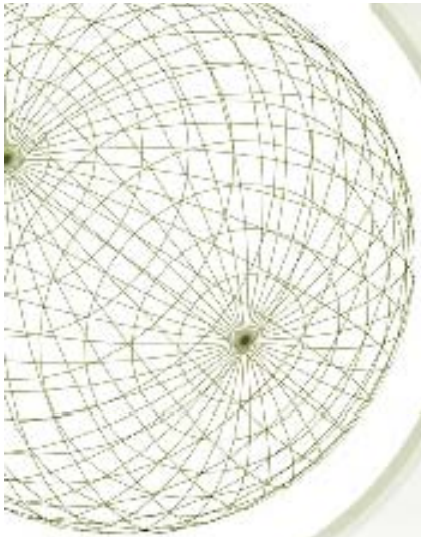


$$E = m_o c^3 \left(\frac{1}{\sqrt{c^2 - v^2}} - \frac{1}{\sqrt{c^2}} \right)$$

$$E = m_o c^3 \left(\frac{1}{\sqrt{c^2 - v^2}} - \frac{1}{c} \right)$$

$$E = m_o c^3 \left(\frac{c}{c\sqrt{c^2 - v^2}} - \frac{\sqrt{c^2 - v^2}}{c\sqrt{c^2 - v^2}} \right)$$

$$E = m_o c^3 \left(\frac{c - \sqrt{c^2 - v^2}}{c\sqrt{c^2 - v^2}} \right)$$

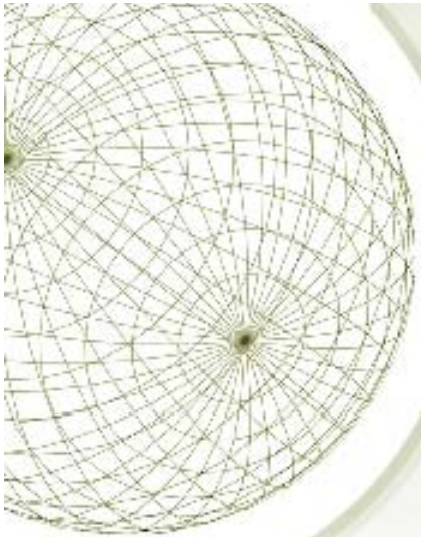


$$E = m_o c^3 \left(\frac{c - \sqrt{c^2 - v^2}}{c\sqrt{c^2 - v^2}}\right)$$

$$E = m_o c^2 \left(\frac{c - \sqrt{c^2 - v^2}}{\sqrt{c^2 - v^2}}\right)$$

$$E = m_o c^2 \left(\frac{c}{\sqrt{c^2 - v^2}} - \frac{\sqrt{c^2 - v^2}}{\sqrt{c^2 - v^2}}\right)$$

$$E = m_o c^2 \left(\frac{c}{\sqrt{c^2 - v^2}} - 1\right)$$

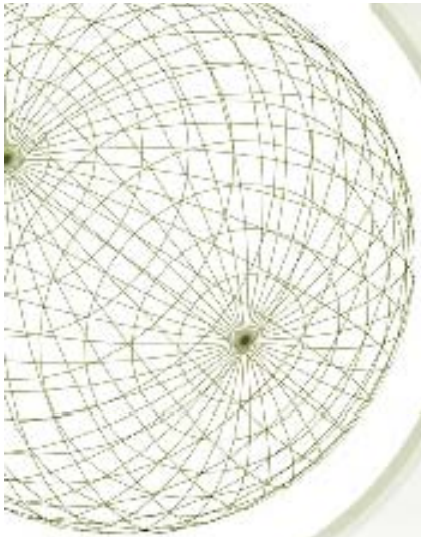


$$E = m_o c^2 \left(\frac{c}{\sqrt{c^2 - v^2}} - 1 \right)$$

$$E = m_o c^2 \left(\frac{1}{\frac{1}{c}\sqrt{c^2 - v^2}} - 1 \right)$$

$$E = m_o c^2 \left(\frac{1}{\sqrt{\left(\frac{1}{c^2}\right)(c^2 - v^2)}} - 1 \right)$$

$$E = m_o c^2 \left(\frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} - 1 \right)$$



$$E = m_o c^2 \left(\frac{1}{\sqrt{(1 - \frac{v^2}{c^2})}} - 1 \right)$$

$$E = c^2 \left(\frac{m_0}{\sqrt{(1 - \frac{v^2}{c^2})}} - m_0 \right)$$

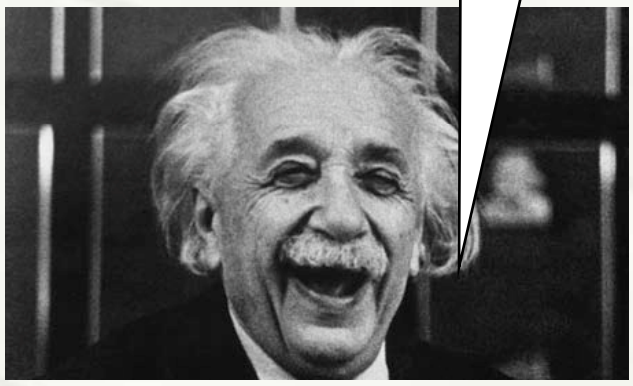
Recall the relativistic mass: $m = \frac{m_0}{\sqrt{(1 - \frac{v^2}{c^2})}}$

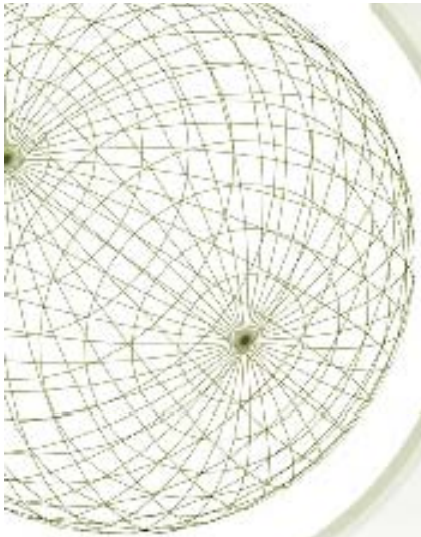
Substituting this yields: $E = c^2 (m - m_0)$

or $\boxed{E = (m - m_o)c^2}$

$$E = mc^2$$

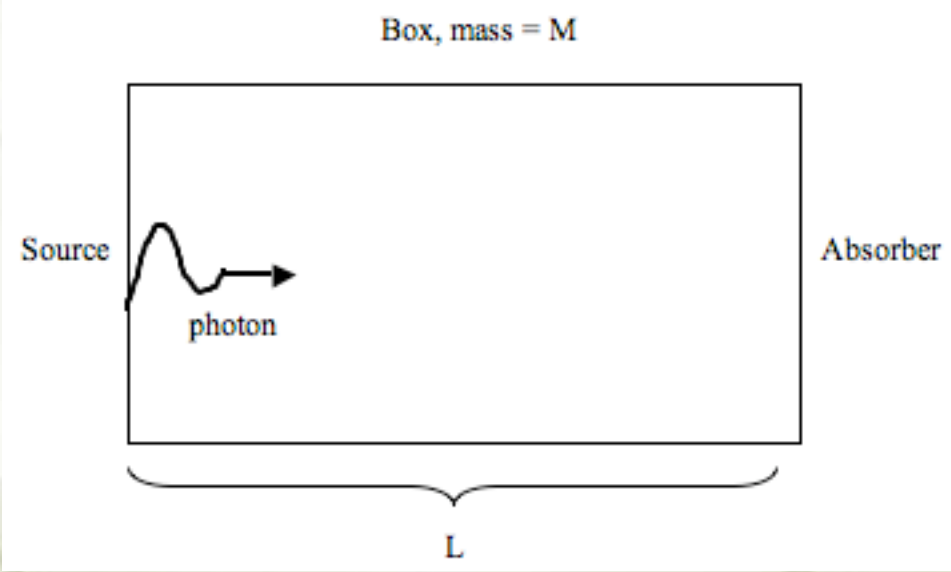
Make everything as simple as possible, but *not* simpler!





Second Derivation

Consider a box with a light source of negligible mass at the left end and a black absorber of negligible mass at the other end. The total mass of the box, source, and absorber is M. a Photon is emitted from the left. It strikes the absorber and is absorbed.



By conservation of momentum, when the photon leaves the source, and travels right, the box must move to the left to maintain a total momentum of 0 in the system. We will use two methods to calculate how far the box moves, and equate the results.

The energy of a photon is given by:

$$E = hf = \frac{hc}{\lambda}$$

From de Broglie's Equation:

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

Thus:

$$E = \frac{hc}{\frac{h}{mv}}$$

$$E = cmv$$

$$\frac{E}{c} = mv = p$$

By the Law of Conservation of Momentum, the momentum of the photon equals the momentum of the box

$$p = MV = \frac{E}{c}$$

$$V = \frac{E}{Mc}$$

From its definition, the velocity of the photon is given by the displacement divided by the time

$$c = \frac{L}{t}$$

$$t = \frac{L}{c}$$

The velocity of the box is: $V = \frac{E}{Mc}$

The amount of time that the box moves is: $t = \frac{L}{c}$

The distance the box moves is:

$$d = Vt$$

$$d = \frac{E}{Mc}\frac{L}{c}$$

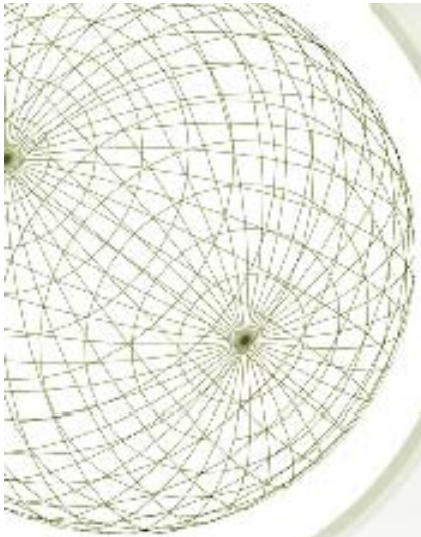
$$d = \frac{EL}{Mc^2}$$

From the mass-energy equivalence, the photon has mass since it has energy. Let this mass be m (m<<M).

Since there is no external force on the system, the center of mass must remain in the same place. Therefore the moments of the box and photon must have the same value:

$$Md = mL$$

$$d = \frac{mL}{M}$$



Equating the two expressions for *d* yields:

$$d = \frac{mL}{M} = \frac{EL}{Mc^2}$$

$$m = \frac{E}{c^2}$$

$$\boxed{E = mc^2}$$